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COMPUTATIONAL VISCOELASTICITY OF AGING MATERIALS

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Abstract. *This study presents a theoretical framework for the computational modelling of viscoelastic materials with a time-dependent, but stress- and strain-independent variation of stiffness, like e.g. solidifying concrete. It offers the possibility to see solidification and deterioration from a common viewpoint instead of applying two separate theories.*

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1 Definition of *Aging*

This paper deals with nonlinear elastic constitutive relations for aging materials. These elastic models are expanded by linear viscoelastic models for time dependent behaviour, especially with concrete in mind. Young hardening concrete is one example for the attribute *aging* as it is used in this paper. Aging is understood here as a time dependent variation of the isotropic homogenous stiffness, which is not caused by stresses. Accordingly the solidification of concrete is an example for increasing stiffness; the damage caused by melting, temperature effects in general or time-deteriorating effects are examples for decreasing stiffness. Any effect that is stress dependent as plasticity or cracking must not be included in this definition because it is not isotropic. It should be mentioned, that anisotropic aging effects might be included into the presented theory by higher order solidification or damage tensors. But in this paper it is assumed, that the change of the time dependent stiffness may be described by scalars according to the following formula:

$$E(t) = \beta(t)[1 - \delta(t)]\bar{E} \tag{1}$$

In this definition $\bar{E}$ is the Young's modulus, that would be reached theoretically at infinite time without any deteriorating effects ($\beta = 1$ and $\delta = 0$). $\beta(t)$ is the degree of solidification, which should not be confused with the degree of hydration $\alpha(t)$. A relation between α and β was e.g. found by Laube [9]:

$$\beta(\alpha) = \frac{E(\alpha(t))}{\bar{E}} = \left(\frac{\alpha(t) - \alpha_o}{1 - \alpha_o}\right)^{2/3} \geq 0 \tag{2}$$

where α_o is the degree of hydration, when mechanical properties start to develop. $\delta(t)$ is the degree of deterioration, that is called D in the classical damage theory of Kachanov [8]. Both degrees β and δ are varying between 0 and 1 and are increasing monotonously with time:

$$\beta \in (0, 1] \quad \text{and} \quad \dot{\beta} \geq 0 \quad ; \quad \delta(t) \in [0, 1) \quad \text{and} \quad \dot{\delta} \geq 0 \tag{3}$$

Especially the monotonous increase is an important property. It indicates, that regaining stiffness after an early deterioration is not decreasing the degree of deterioration but increasing the degree of solidification. This is fundamental, because solidification is not the reverse process of deterioration as will be shown later on.

It is assumed that all conditions are satisfied for an additive decomposition of the strain rates in parts caused by (in)elasticity (e), viscousity (φ), temperature (ϑ), shrinkage (s) and cracking (c):

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^\varphi + \boldsymbol{\varepsilon}^\vartheta + \boldsymbol{\varepsilon}^s + \boldsymbol{\varepsilon}^c \tag{4}$$

2 Nonlinear Elasticity of Aging Materials

The difference between an increasing and a decreasing stiffness is shown by a simple example, that everyone knows, who has butter for breakfast: From observation we know, that the heating of butter leads to a loss of stiffness and to an increasing deformation i.e. strains (if we imagine some kind of loading on the butter and neglect plasticity effects). If the deformed butter is put back in the refrigerator and the temperature is cooling down, the mechanical deterioration process is not reversed. On the contrary no deformation at all is observed during solidification. The butter hardens in its deformed state and doesn't regain it's original shape. In this context concrete behaves like butter.

This may also be explained by a graphical model that was introduced by Bažant with his solidification theory for concrete [2]. One can imagine a solidification process as the increase of the solidified share of a body. An increase of the degree of solidification means that new stress-free connections are built (Fig. 1). The opposite process of deterioration may be likewise imagined as the damage of the already existing solidified part, that already bears stresses. In an equivalent model we may think of deterioration as cutting stressed springs (what leads to deformation) and of solidification as introducing new stress-free springs (what does not cause any deformation).

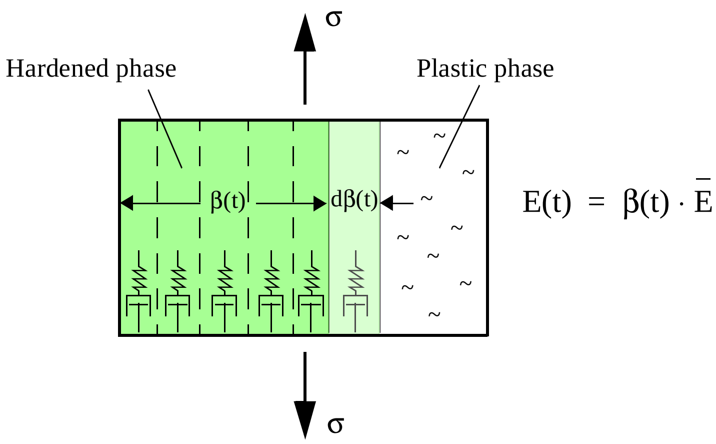


Figure 1: Solidification Theory [2].

For doing some analysis on aging materials, we have to formulate constitutive equations, which reflect this difference. Because both phenomena have a transient nature, a rate type formulation is chosen. For pure solidification and pure deterioration the constitutive equations are well known:

$$\text{Solidification}(\dot{\delta} = 0): \quad \dot{\varepsilon}^e = \frac{\dot{\sigma}}{E} \quad \text{(equivalent to the solidification theory)} \tag{5}$$

$$\text{Deterioration}(\dot{\beta} = 0): \dot{\varepsilon}^e = \frac{\partial}{\partial t}(\sigma / E) \quad \text{(equivalent to Kachanov's damage theory)} \tag{6}$$

As is easily seen, the formula for solidification leads to changes of the strains only through changes of the stress state. That means solidification alone doesn't affect strains or stresses. From the formula for deterioration follows, that a change of E may cause a change of the strains too. These nonlinear-elastic strains are often called inelastic because they are not reversible. In this paper, they are included in the (nonlinear-) elastic strain ε^e.

The generalization of the presented one dimensional equation is done by replacing E with the elastic stiffness tensor. We use a matrix formulation and introduce **D** as the inverse matrix of the material stiffness **E**:

$$\mathbf{D} = \mathbf{E}^{-1} \tag{7}$$

The equations given above in multidimensional form then are:

$$\text{Solidification } (\dot{\delta} = 0): \quad \dot{\boldsymbol{\varepsilon}}^e = \mathbf{D}\dot{\boldsymbol{\sigma}} \tag{8}$$

$$\text{Deterioration } (\dot{\beta} = 0): \quad \dot{\boldsymbol{\varepsilon}}^e = \frac{\partial}{\partial t}(\mathbf{D}\boldsymbol{\sigma}) = \mathbf{D}\dot{\boldsymbol{\sigma}} + \dot{\mathbf{D}}\boldsymbol{\sigma} \tag{9}$$

As far as the authors are aware, there is no theory for the combination of both effects taking place simultanously. While butter can't be heated and cooled simultaneously and such a theory isn't needed for that problem, there are materials that show solidification or deterioration for other reasons. E.g. concrete hardening is caused by a chemical exothermal reaction, that is sometimes taking place under very high temperatures, so that indeed a combined solidification and deterioration process exists. A combined theory offers also the advantage of providing only one constitutive relation and therefore one algorithm for both problems.

Under this aspects, we developed a single generalized theory for aging materials. The elastic behaviour of cutting springs and building new stress-free springs is described by applying only the derivative with respect to the degree of deterioration:

$$\dot{\boldsymbol{\varepsilon}}^e = \mathbf{D}\dot{\boldsymbol{\sigma}} + \frac{\partial \mathbf{D}}{\partial \delta}\dot{\delta}\boldsymbol{\sigma} = \mathbf{D}\dot{\boldsymbol{\sigma}} + \frac{\dot{\delta}}{1-\delta}\mathbf{D}\boldsymbol{\sigma} \tag{10}$$

Without solidification effects this equation is equivalent to Kachanov's damage theory [8], while it also incooperates the pure solidification theory [2]. A summary of aging elasticity for a single stress change in time is given in table 1.

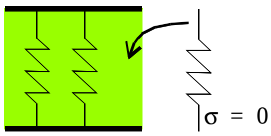
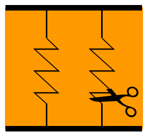
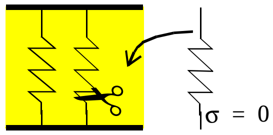
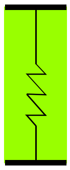
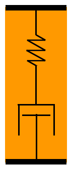
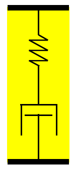
	Solidification	*Deterioration*	*Solidification with Deterioration*
Mechanical Model			
Young's Modulus	$E(t) = \beta(t)E$	$E(t) = [1 - \delta(t)]E$	$E(t) = \beta(t)[1 - \delta(t)]E$
Rheological Model			
Instantanous elastic strain	$\boldsymbol{\varepsilon}^e(t_o) = \mathbf{D}(t_o)\boldsymbol{\sigma}(t_o)$		
Inelastic strain rate	$\dot{\boldsymbol{\varepsilon}}^e(t) = 0$	$\dot{\boldsymbol{\varepsilon}}^e(t) = \dot{\mathbf{D}}(t)\boldsymbol{\sigma}(t_o)$	$\dot{\boldsymbol{\varepsilon}}^e(t) = \dfrac{\dot{\delta}(t)}{1 - \delta(t)}\mathbf{D}(t)\boldsymbol{\sigma}(t_o)$

Table 1: Aging elasticity for a one-time stress change $\sigma(t_o)$.

In a simple example the unidimensional cases of pure deterioration, pure solidification and the combination are compared in fig. 2.

3 Linear Viscoelasticity

3.1 Goals of a numerical model

The viscoelastic theory that is presented below was derived for concrete, although it may be applicable to other materials too. The viscous behaviour of concrete can be described through linear viscoelasticity, if stress levels are below 40 % of the compressive strength. Therefore linear viscoelasticity is appropriate for almost all concrete structures because the stress criterion is satisfied under permanent or frequent loads. The goal of the viscoelastic material law is to give a numerical representation of the measured viscoelastic behaviour. For concrete the viscoelastic properties are given in form of creep functions φ. A numerical model should be formulated with respect to φ because no other material parameters are usually available for concrete. For the sake of simplicity we presume affinity of the creep strains, i.e. the creep strains are supposed to occur with the same Poisson's ratio ν as the elastic strains (some authors suggest volumetrically constant creep for concrete [7]).

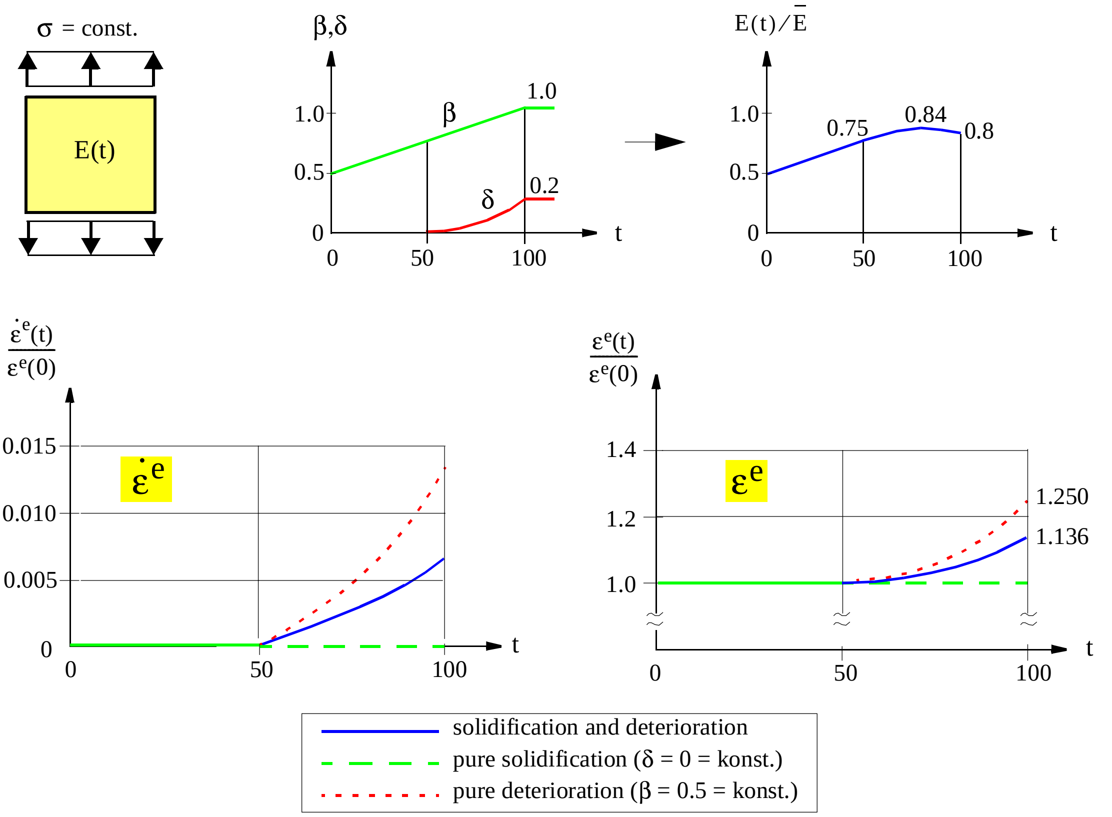


Figure 2: Example for aging elasticity (combined solidification and deterioration).

3.2 Creep functions

The starting point of a viscoelastic analysis is a good creep function. What good means in this context depends as always on the point of view. For some materials like polymers, a definition of φ depending on the load duration time ($t-t_o$) is good, because these materials don't show an aging creep behaviour (nevertheless they may be aging). When Dischinger first formulated a rate type constitutive equation for concrete creep in 1937, he presumed the so called Withney ideal creep curves, that lead him to the rate of creep method. Nowadays most authors agree that this model isn't good enough for concrete. Only creep functions that depend on two variables – the actual time t and the time t_o where the creep inducing stress change occurred – are supposed to be accurate. Because one may argue about this point, we will present a numerical model, that is based on the most general formulation of $\varphi(t,t_o)$ with two independent variables t and t_o. The creep function itself is defined as the ratio between the viscous strain and the corresponding elastic strain without deterioration effects:

$$\varphi(t, t_o) = \frac{\varepsilon^{\varphi}(t, t_o)}{\varepsilon^{e}(t_o)} \tag{11}$$

There also exist other definitions. Especially in most modern concrete codes a definition is used, that relates the creep strain to a virtual elastic strain, that the creep inducing stress would cause at the concrete age of 28 days:

$$\varphi_{E28}(t, t_o) = \frac{\varepsilon^{\varphi}(t, t_o)}{\varepsilon^{e}_{E28}} = \frac{\varepsilon^{\varphi}(t, t_o)}{\sigma(t_o)/E_{28d}} \tag{12}$$

It should be mentioned here, that this definition leads to other constitutive equations than the ones presented later on. Fortunately there is a conversion rule:

$$\varphi_{E28}(t, t_o) = \frac{E_{28}}{E(t_o)}\varphi(t, t_o) \tag{13}$$

The difference has to be considered too, when using an effective Young's modulus for creep representation:

$$\text{Without solidification:}\quad E_{eff} = \frac{E(t_o)}{1 + \varphi_{E28}(t_1, t_o)} \quad \text{(MC90, EC2)} \tag{14}$$

$$\text{With solidification:}\quad E_{eff} = \frac{E(t_o)}{1 + \varphi(t_1, t_o)} = \frac{E(t_o)}{1 + \frac{E(t_o)}{E_{28}}\varphi_{E28}(t_1, t_o)} \tag{15}$$

Model Code 90 and EC2 don't consider this difference in the effective stiffness (eq. 14). The equation given there is only applicable to old, i.e. hardened concrete at loading time.

Analytical creep functions were proposed by many authors, most of them for mature concrete [5]. If young solidifying concrete is analysed, a creep function should be used, that is well suited for young concrete as the one that Laube [9] proposed.

4 Viscoelasticity for aging materials

4.1 Rate type formulation

A rate type formulation of the constitutive equations will be used and an additive separation of the viscous and elastic part is assumed:

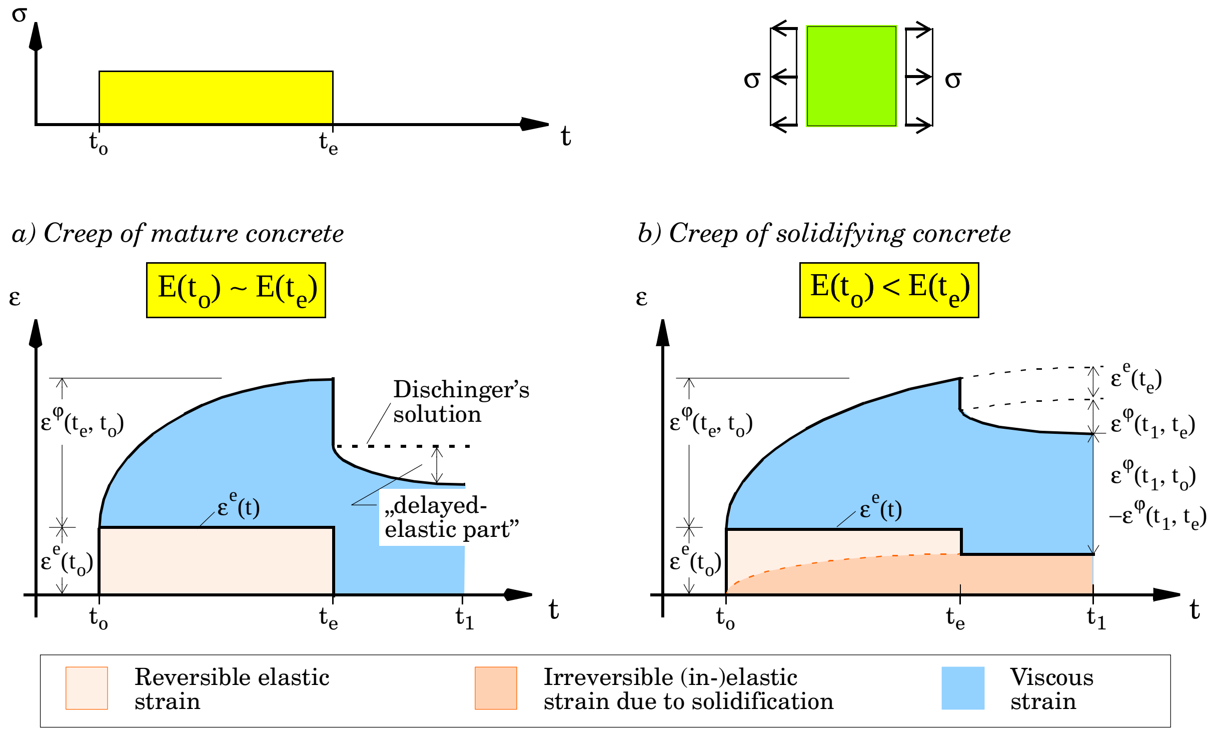


Figure 3: Viscoelastic behaviour: a) without, b) with the consideration of aging.

$$\dot{\varepsilon}^{e+\varphi} = \dot{\varepsilon}^{e} + \dot{\varepsilon}^{\varphi} \tag{16}$$

To find a mathematical model that is consistent with the definition of the creep function, one has to consider the full time derivative:

$$\dot{\boldsymbol{\varepsilon}}^{\varphi} = \frac{\partial}{\partial t}[\varphi \mathbf{D}\boldsymbol{\sigma}] = \underbrace{\dot{\varphi}\mathbf{D}\boldsymbol{\sigma}}_{\dot{\boldsymbol{\varepsilon}}_1^{\varphi}} + \underbrace{\varphi\dot{\mathbf{D}}\boldsymbol{\sigma}}_{\dot{\boldsymbol{\varepsilon}}_2^{\varphi}} + \underbrace{\varphi\mathbf{D}\dot{\boldsymbol{\sigma}}}_{\dot{\boldsymbol{\varepsilon}}_3^{\varphi}} \tag{17}$$

For nonaging materials the second part $\dot{\boldsymbol{\varepsilon}}_2^{\varphi}$ vanishes. For aging materials especially this second part must be investigated more closely. Therefore in the following chapter all three parts of the viscous strain rate, that where already identified by Huckfeldt [7] are explained.

4.2 Viscous strains for solidification and deterioration

We now look at the three parts of the viscous strain rate as introduced above. The first part is the standard or primary term of creep. It's the sole part if stiffness and stresses are constant, and it's integration over time results in the creep function itself. So this is the only part, that can be verified by standard creep tests. For pure solidification and deterioration the term $\dot{\boldsymbol{\varepsilon}}_1^{\varphi}$ is for a constant stress $\boldsymbol{\sigma}_o(t_o)$:

$$\dot{\boldsymbol{\varepsilon}}_1^{\varphi}(t) = \dot{\varphi}(t, t_o)\mathbf{D}(t_o)\boldsymbol{\sigma}_o \tag{18}$$

If the stresses vary over time, one has to integrate over all loading times τ:

$$\dot{\boldsymbol{\varepsilon}}_1^{\varphi}(t) = \int_0^t \dot{\varphi}(t, \tau)\mathbf{D}(\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau \tag{19}$$

The second part $\dot{\boldsymbol{\varepsilon}}_2^{\varphi}$ is creep, that is caused by the rate of the stiffness corresponding to the second part of the elastic strain rate. For solidification this part must not be considered (cf. eq. 8).

$$\text{Solidification:} \quad \frac{\partial}{\partial t}\mathbf{D}(t_o) = 0 \quad \Rightarrow \quad \dot{\boldsymbol{\varepsilon}}_2^{\varphi} = 0 \tag{20}$$

For deterioration the inelastic strain rates $\dot{\boldsymbol{\varepsilon}}_2^{e}$, which are caused by damage at time $\tilde{t}$, are followed by new creep strains $\boldsymbol{\varepsilon}_2^{\varphi}$:

$$\text{Deterioration:} \quad \boldsymbol{\varepsilon}_2^{\varphi}(t) = \int_{t_o}^t \varphi(t, \tilde{t})\dot{\boldsymbol{\varepsilon}}_2^{e}(\tilde{t})d\tilde{t} = \boldsymbol{\sigma}_o(t_o)\int_{t_o}^t \varphi(t, \tilde{t})\dot{\mathbf{D}}(\tilde{t})d\tilde{t} \tag{21}$$

The strain rate is obtained according to the rules for the derivation of integral equations of the Volterra type:

$$\dot{\boldsymbol{\varepsilon}}_2^{\varphi}(t) = \boldsymbol{\sigma}_o(t_o)\varphi(t, t)\dot{\mathbf{D}}(t) + \boldsymbol{\sigma}_o(t_o)\int_{t_o}^t \dot{\varphi}(t, \tilde{t})\dot{\mathbf{D}}(\tilde{t})d\tilde{t} \tag{22}$$

For continous stress changes eq. (22) must be integrated over all loading times τ:

$$\dot{\boldsymbol{\varepsilon}}_2^{\varphi}(t) = \underbrace{\int_0^t \dot{\boldsymbol{\sigma}}(\tau)\varphi(t, t)\dot{\mathbf{D}}(t)d\tau}_{\boldsymbol{\sigma}(t)\varphi(t, t)\dot{\mathbf{D}}(t)} + \int_0^t \dot{\boldsymbol{\sigma}}(\tau)\left(\int_\tau^t \dot{\varphi}(t, \tilde{t})\dot{\mathbf{D}}(\tilde{t})d\tilde{t}\right)d\tau \tag{23}$$

The third part $\dot{\boldsymbol{\varepsilon}}_3^{\varphi}$ is the instantaneous creep of the stress rate $\dot{\boldsymbol{\sigma}}$. Together with the first part of the elastic strain rate one gets the instantaneous strain rate:

$$\dot{\boldsymbol{\varepsilon}}^{e} + \dot{\boldsymbol{\varepsilon}}_3^{\varphi} = (1 + \varphi)\mathbf{D}\dot{\boldsymbol{\sigma}} \tag{24}$$

As normally the whole instantaneous deformation is considered by the Young's modulus, $\varphi(t,t_o{=}t)$ usually is zero. Therefore $\dot{\boldsymbol{\varepsilon}}_3^{\varphi}$ as well as the first term in eq. (23) are zero in the differential form. But they must be considered when transformation to the incremental form of the constitutive equation is carried out:

$$\Delta\boldsymbol{\varepsilon}_3^{\varphi} = \int_{t_i}^{t_{i+1}} \varphi(t_{i+1}, \tau)\mathbf{D}(\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau \tag{25}$$

4.3 Viscoelasticity for deterioration and solidification

Now one may combine the different parts to a viscoelastic constitutive law. First the case of pure deterioration is given by eq. 26:

Deterioration $\dot{E} \le 0$

$$\begin{aligned}
\dot{\boldsymbol{\varepsilon}}^{e}(t) + \dot{\boldsymbol{\varepsilon}}^{\varphi}(t) = {} & \mathbf{D}(t)\dot{\boldsymbol{\sigma}}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}_1^{e} \\
& + \dot{\mathbf{D}}(t)\boldsymbol{\sigma}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}_2^{e} \\
& + \int_0^t \dot{\varphi}(t,\tau)\mathbf{D}(\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau && \rightarrow \dot{\boldsymbol{\varepsilon}}_1^{\varphi} \\
& + \boldsymbol{\sigma}(t)\varphi(t,t)\dot{\mathbf{D}}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}_{2a}^{\varphi} \\
& + \int_0^t \boldsymbol{\sigma}(\tau)\left(\int_\tau^t \dot{\varphi}(t,\tilde{t})\dot{\mathbf{D}}(\tilde{t})d\tilde{t}\right)d\tau && \rightarrow \dot{\boldsymbol{\varepsilon}}_{2b}^{\varphi} \\
& + \mathbf{D}(t)\varphi(t,t)\dot{\boldsymbol{\sigma}}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}_3^{\varphi}
\end{aligned} \tag{26}$$

The case of pure solidification without deteriorating effects is obtained by omitting the second parts of the elastic and viscous strain rates. Eq. (27) doesn't depend on the rate of stiffness, according to the idea, that an increasing stiffness doesn't affect the strains directly. The influence of solidification on the creep process is included automatically in the creep function itself, wich is derived from experiments with hardening concrete specimen (aging creep function).

Solidification: $\dot{E} \geq 0$

$$\begin{aligned}
\dot{\boldsymbol{\varepsilon}}^{e}(t) + \dot{\boldsymbol{\varepsilon}}^{\varphi}(t) \;=\; & \mathbf{D}(t)\dot{\boldsymbol{\sigma}}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}^{e}_{1} \\
& + \int_{0}^{t} \dot{\varphi}(t,\tau)\mathbf{D}(\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau && \rightarrow \dot{\boldsymbol{\varepsilon}}^{\varphi}_{1} \\
& + \mathbf{D}(t)\varphi(t,t)\dot{\boldsymbol{\sigma}}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}^{\varphi}_{3}
\end{aligned} \tag{27}$$

The derived formulations for pure solidification and deterioration don’t allow the analysis of problems, where both phenomena occur together. Furthermore for algorithmic reasons it is inconvenient to distinguish between two cases. Thus we derived a unified theory by using the same general representation of the stiffness as in chapter 2:

$$E(t) \;=\; \beta(t)[1 - \delta(t)]\bar{E} \tag{28}$$

For the unified approach we now must presume, that the material shows a common creep behaviour for increasing and decreasing stiffness, that is again defined by a creep function according to eq. (11). From this definition we get the viscous strain caused by a constant stress $\boldsymbol{\sigma}_o(t_o)$:

$$\begin{aligned}
\boldsymbol{\varepsilon}^{\varphi}(t) \;&=\; \varphi(t,t_o)\boldsymbol{\varepsilon}^{e}_{1}(t_o) + \int_{t_o}^{t} \varphi(t,\tilde{t})\dot{\boldsymbol{\varepsilon}}^{e}_{2}(\tilde{t})d\tilde{t} \;= \\
&=\; \varphi(t,t_o)\mathbf{D}(t_o)\boldsymbol{\sigma}_o(t_o) + \int_{t_o}^{t} \varphi(t,\tilde{t})\frac{\dot{\delta}(\tilde{t})}{1-\delta(\tilde{t})}\mathbf{D}(\tilde{t})\boldsymbol{\sigma}_o(t_o)d\tilde{t}
\end{aligned} \tag{29}$$

Considering the demonstrative model of including and cutting springs (table 1), it is found that the rate type constitutive law again is obtained by applying the time derivative only to the deteriorating part of the stiffness change:

$$\dot{\mathbf{D}}(t) \rightarrow \frac{\partial \mathbf{D}(\beta,\delta)}{\partial \delta}\dot{\delta} \;=\; \frac{\dot{\delta}(t)}{1-\delta(t)}\mathbf{D}(t) \tag{30}$$

So we finally get a unified viscoelastic constitutive law in rate type form:

$$
\begin{aligned}
\dot{\boldsymbol{\varepsilon}}^{e}(t) + \dot{\boldsymbol{\varepsilon}}^{\varphi}(t) \;=\; & \mathbf{D}(t)\dot{\boldsymbol{\sigma}}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}^{e}_{1} \\
& + \frac{\dot{\delta}(t)}{1-\delta(t)}\mathbf{D}(t)\boldsymbol{\sigma}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}^{e}_{2} \\
& + \int_0^t \dot{\varphi}(t,\tau)\mathbf{D}(\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau && \rightarrow \dot{\boldsymbol{\varepsilon}}^{\varphi}_{1} \\
& + \varphi(t,t)\frac{\dot{\delta}(t)}{1-\delta(t)}\mathbf{D}(t)\boldsymbol{\sigma}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}^{\varphi}_{2a} \\
& + \int_0^t \dot{\boldsymbol{\sigma}}(\tau)\left(\int_\tau^t \dot{\varphi}(t,\tilde{t})\frac{\dot{\delta}(\tilde{t})}{1-\delta(\tilde{t})}\mathbf{D}(\tilde{t})d\tilde{t}\right)d\tau && \rightarrow \dot{\boldsymbol{\varepsilon}}^{\varphi}_{2b} \\
& + \varphi(t,t)\mathbf{D}(t)\dot{\boldsymbol{\sigma}}(t) && \rightarrow \dot{\boldsymbol{\varepsilon}}^{\varphi}_{3}
\end{aligned}
\qquad (31)
$$

4.4 Example

Again the simple example from fig. 2 is chosen to show the different parts of the viscous strain rate. $\dot{\boldsymbol{\varepsilon}}^{\varphi}_{2a}$ and $\dot{\boldsymbol{\varepsilon}}^{\varphi}_{3}$ are zero according to the starting value of the creep function and the rate type formulation. If large time steps within an incremental algorith are used, these parts may reach a noteworthy magnitude. The influence of damage on the creep strain is shown in fig. 4 by comparision with the strain of the pure solidification problem.

5 Computational Realization

5.1 Incremental Form

According to the rate type formulation the incremental change of the inverse material stiffness **D** is obtained by the consideration of deteriorating effects only:

$$
\Delta\mathbf{D} \;=\; \mathbf{D}(t_{i+1}) - \mathbf{D}(t_i) \rightarrow \left(\frac{1}{1-\delta_{i+1}} - \frac{1}{1-\delta_i}\right)\frac{\mathbf{F}}{\beta_{i+1}\bar{\bar{E}}} = \frac{\Delta\delta}{1-\delta(t_i)}\mathbf{D}(t_{i+1}) \qquad (32)
$$

The components of the matrix **F,** which gives the multidimensional form of the material stiffness, are defined by

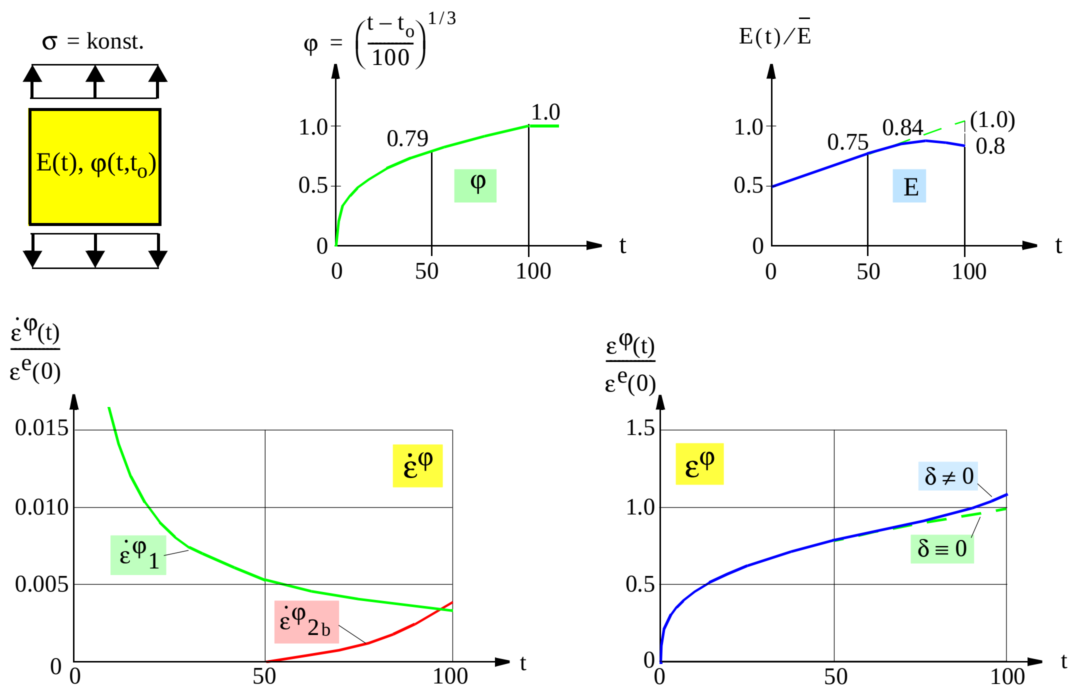


Figure 4: Example for aging viscosity (combined solidification and deterioration).

$$F_{ijkl} = \frac{1}{2}(1+\nu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \nu\delta_{ij}\delta_{kl} \ , \tag{33}$$

where ν is the Poisson’s ration, which is assumed to be independent of time, and δ_{ij} is the Kronecker delta.

Thereof the generalized incremental constitutive law for the elastic strain rates is:

$$\Delta\boldsymbol{\varepsilon}^{e} = \int_{0}^{t_{i+1}} \mathbf{D}(t_{i+1})\dot{\boldsymbol{\sigma}}(\tau)d\tau - \int_{0}^{t_i} \mathbf{D}(t_i)\dot{\boldsymbol{\sigma}}(\tau)d\tau$$

$$= (\mathbf{D}(t_{i+1}) - \mathbf{D}(t_i))\int_{0}^{t_i}\dot{\boldsymbol{\sigma}}(\tau)d\tau + \mathbf{D}(t_{i+1})\int_{t_i}^{t_{i+1}}\dot{\boldsymbol{\sigma}}(\tau)d\tau$$

$$= \underbrace{\left(\frac{\Delta\delta}{1-\delta(t_i)}\mathbf{D}(t_{i+1})\right)\boldsymbol{\sigma}(t_i)}_{\Delta\boldsymbol{\varepsilon}^{e}_{2}} + \underbrace{\mathbf{D}(t_{i+1})\Delta\boldsymbol{\sigma}}_{\Delta\boldsymbol{\varepsilon}^{e}_{1}} = \mathbf{D}(t_{i+1})\left(\frac{\Delta\delta}{1-\delta(t_i)}\boldsymbol{\sigma}(t_i) + \Delta\boldsymbol{\sigma}\right) \tag{34}$$

For the implementation in an FE algorithm the inverse relation is required:

$$\Delta\boldsymbol{\sigma} = \mathbf{D}(t_{i+1})^{-1}\left(\Delta\boldsymbol{\varepsilon}^{e} - \frac{\Delta\delta}{1-\delta(t_i)}\mathbf{D}(t_{i+1})\boldsymbol{\sigma}(t_i)\right) = \underbrace{\mathbf{D}(t_{i+1})^{-1}}_{\mathbf{E}(t_{i+1})}\Delta\boldsymbol{\varepsilon}^{e} - \underbrace{\frac{\Delta\delta}{1-\delta(t_i)}\boldsymbol{\sigma}(t_i)}_{\rightarrow \Delta\mathbf{P}_{int}} \qquad (35)$$

The increase of the load vector $\Delta\mathbf{P}_{int}$ represents the stresses, that are set free due to the increase of deterioration within one time step.

To get the incremental viscous strains, eq. 29 is transformed and an abbreviation $\tilde{\mathbf{D}}(t, \tau)$ is introduced:

$$\boldsymbol{\varepsilon}^{\varphi}(t) = \int_0^t \dot{\boldsymbol{\sigma}}(\tau)\underbrace{\left(\varphi(t,\tau)\mathbf{D}(\tau) + \int_\tau^t \varphi(t,\tilde{t})\frac{\dot{\delta}(\tilde{t})}{1-\delta(\tilde{t})}\mathbf{D}(\tilde{t})d\tilde{t}\right)}_{\tilde{\mathbf{D}}(t,\tau)}d\tau \qquad (36)$$

The increment of the viscous strain is then:

$$\Delta\boldsymbol{\varepsilon}^{\varphi} = \int_0^{t_{i+1}} \tilde{\mathbf{D}}(t_{i+1},\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau - \int_0^{t_i}\tilde{\mathbf{D}}(t_i,\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau \qquad (37)$$

$$= \int_0^{t_i}[\tilde{\mathbf{D}}(t_{i+1},\tau) - \tilde{\mathbf{D}}(t_i,\tau)]\dot{\boldsymbol{\sigma}}(\tau)d\tau + \int_{t_i}^{t_{i+1}} \tilde{\mathbf{D}}(t_{i+1},\tau)\dot{\boldsymbol{\sigma}}(\tau)d\tau$$

5.2 Integration of the stress history

The presented constitutive relations require the evaluation of integrals over the whole stress and stiffness history. This leads to remarkable computing times for larger systems. Not long ago, this kind of calculation simply wasn't possible and a work around was needed for similar kinds of constitutive equations. An elegant way was introduced by Bažant [2] that solves the problem by the evaluation of a Dirichlet series. Another solution is the *rate of creep method* that demands special creep functions (cf. chapter 3.2). But if the computing power continues to double according to Moores law every 18 month, the real evaluation of the history integrals will soon be possible for large systems with reasonable computational effort.

To circumvent the storage problem, there is a work around for pure solidification problems: It is possible not to save the history of the stresses, but the strains for every time step in the future. In the actual time step i, the viscous strain increment for all future time steps j > i are calculated and added to the values already inherited from former time steps. This is advantageous, because the storage effort is a maximum at the beginning of the calculation. During the calculation the storage space of already evaluated time steps may be set free and used for the increasing storage requirements for the results. A drawback of this method is, that the sequence of time steps can't be adapted during the calculation and that a continuation of a finished calculation isn't possible.

6 Conclusion

There is a significant difference in the behaviour of solidifying i.e. time-hardening materials like concrete, and melting materials, that can be seen in this context as a representation of damage due to non−mechanical parameters (time, temperature). In a first approach both phenomena can be described by a time−dependent variation of their elastic stiffness, assuming a constant Poisson's ratio (eq. 1). Presently two separate theories are used to model the physical behaviour for increasing and decreasing E. In this paper the shared and the different theoretical parts in the context of a unified viscoelastic constitutive model are discussed, where the elastic and the viscous parts of the strain rates are decomposed.

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